Lecture abstract

Topics covered in this presentation

▶ Constrained Optimal Control for Linear Discrete-Time Systems
▶ Basic Formulation of a Model Predictive Control Regulation Problem
▶ How to Ensure Feasibility and Stability
▶ Examples

We will not have time to cover the following

▶ Theory of Convex Optimization
▶ Background on Set-Theoretic Methods in Control
▶ Proofs (stability, feasibility)
▶ Details of Real-Time Implementation
▶ Extensions (tracking problems, effect of uncertainties, etc.)
Chapter outline

- Introduction to Model Predictive Control
  - 1 Introduction
  - 2 Discrete Time Constrained Optimal Control
  - 3 The Basic MPC Formulation
  - 4 Ensuring Feasibility and Stability
  - 5 Example: Constrained Double Integrator
  - 6 Extensions and Generalizations
Introduction to Model Predictive Control

1 Introduction

2 Discrete Time Constrained Optimal Control

3 The Basic MPC Formulation

4 Ensuring Feasibility and Stability

5 Example: Constrained Double Integrator

6 Extensions and Generalizations
What is Model Predictive Control?

Properties of Model Predictive Control (MPC)

- model-based
- prediction of future system behavior
- built-in constraint satisfaction (hard constraints)
- general formulation for MIMO systems

Applications of Model Predictive Control

- traditionally: process industry (since the 1980’s)
- high-performance control applications
- automotive industry
- building automation and control
- still dwarfed by PID in industry, but growing
PID vs. MPC

\[ \frac{dx}{dt} = F(t, x, u), \quad x(0) = x_0, \]

\[ J(u) = \int L(t, x, \dot{x}, u) \rightarrow \text{extr.} \]

Example: Audi Smart Engine

- Design of an MPC Controller regulating the desired speed (through an Automatic Cruise Control) in order to reach the destination in the most fuel-efficient way
- Prediction: Max and Min Speed of traffic, Grade
- Constraints: Max and Min Speed (of traffic and of vehicle)

Example courtesy of Prof. Borrelli
Receding Horizon Control

Given $x(t)$, optimize over finite horizon of length $N$ at time $t$.
Only apply the first optimal move $u(t)$ of the predicted controls.
Shift the horizon by one.
Repeat the whole optimization at time $t + 1$.
Optimization using current measurements $\Rightarrow$ Feedback!
Receding Horizon Control

Given $x(t)$, optimize over finite horizon of length $N$ at time $t$
Receding Horizon Control

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Receding Horizon Control

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- Repeat the whole optimization at time $t + 1$
- Optimization using current measurements ⇒ Feedback!
Important Issues in Model Predictive Control

- **Feasibility**: Optimization problem may become infeasible at some future time step
- **Stability**: Closed-loop stability not guaranteed
- **Performance**: Effect of finite prediction horizon?

\[
\min \sum_{k=t}^{\infty} l(x_k, u_k) \quad \text{vs.} \quad \min \sum_{k=t}^{t+N} l(x_k, u_k) \quad \text{repeatedly}
\]

- **Real-Time Implementation**: Reliable fast computation needed

Note: The above issues arise even when assuming a perfect system model and no disturbances.
Introduction to Model Predictive Control

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Continuous time LQR

\[ J_c = \min_u \int_0^\infty x(t)^T Q x(t) + u(t)^T R u(t) \, dt \]
\[ \text{s.t.} \quad \dot{x}(t) = A x(t) + B u(t) \]

Discrete time LQR

\[ J_d = \min_u \sum_{k=0}^\infty x_k^T Q x_k + u_k^T R u_k \]
\[ \text{s.t.} \quad x_{k+1} = A_d x_k + B_d u_k \]

Why discrete time?

- software implementations of controllers are inherently discrete
- need to discretize sooner or later anyway
How to Deal With Constraints?

In practice, more or less all systems are constrained

▶ input saturation (actuation limits)
  ▶ e.g. limited motor voltage of cart system in the lab
▶ state constraints (physical limits, safety limits)
  ▶ e.g. finite track length of the pendulum system

Constraints

▶ \( u(t) \in U \) for all \( t \), input constraint set \( U \subseteq \mathbb{R}^p \)
▶ \( x(t) \in \mathcal{X} \) for all \( t \), state constraint set \( \mathcal{X} \subseteq \mathbb{R}^n \)

Often assumed in MPC: \( U \) and \( \mathcal{X} \) are polyhedra

▶ makes \( x(t) \in \mathcal{X} \iff H_x x(t) \leq E_x \) and \( u(t) \in U \iff H_u u(t) \leq E_u \) linear constraints

\(^a\) a finite intersection of half-spaces

Issue: No systematic treatment by conventional methods (PID, LQR...)

The Naive Approach: Just Truncate!

Stage problem at time \( t \), given state \( x(t) \)

\[
\begin{align*}
\min_{u_0, \ldots, u_{N-1}} & \quad \sum_{k=0}^{N-1} x_k^T Q x_k + u_k^T R u_k \\
\text{s.t.} & \quad x_{k+1} = A x_k + B u_k, \quad k = 0, \ldots, N - 1 \\
& \quad u_k \in \mathcal{U}, \quad x_{k+1} \in \mathcal{X}, \quad k = 0, \ldots, N - 1 \\
& \quad x_0 = x(t)
\end{align*}
\]

- If \( \mathcal{X} \) and \( \mathcal{U} \) are polyhedra, this is a Quadratic Program
  - can be solved fast, efficiently and reliably using modern solvers

Basic MPC Algorithm:
- at time \( t \), measure state \( x(t) \) and solve stage problem
- apply control \( u(t) = u_0 \)
- go to time \( t + 1 \) and repeat
The Naive Approach: Feasibility Issues

**Double Integrator Example**

\[ A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad R = 10 \]

\[ X = \{ x \in \mathbb{R}^2 : |x_i| \leq 5, \ i = 1, 2 \}, \quad U = \{ u \in \mathbb{R} : |u| \leq 1 \} \]
The Naive Approach: Feasibility Issues

Double Integrator Example

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The Naive Approach: Feasibility Issues

Observations

- LQR controller violates input constraints
- “naive” MPC with prediction horizon $N = 3$ is infeasible at $t = 2$
- “naive” MPC with prediction horizon $N = 4$ stays feasible throughout, but has poor performance
- controller needs to “look ahead far enough” for good performance
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The Basic MPC Formulation

Stage problem at time $t$, given state $x(t)$

$$\begin{align*}
\min_{u_0, \ldots, u_{N-1}} \quad & \sum_{k=0}^{N-1} x_k^T Q x_k + u_k^T R u_k + x_N^T P x_N \\
\text{s.t.} \quad & x_{k+1} = A x_k + B u_k, \quad k = 0, \ldots, N - 1 \\
& u_k \in \mathcal{U}, \quad x_{k+1} \in \mathcal{X}, \quad k = 0, \ldots, N - 1 \\
& x_N \in \mathcal{X}_f \\
& x_0 = x(t)
\end{align*}$$

- terminal weighting matrix $P$
  - used to achieve stability
- terminal constraint set $\mathcal{X}_f$
  - used to achieve persistent feasibility
Terminal cost and Terminal Constraints

**Obvious:** If $\mathcal{X}_f = \{0\}$ and a solution to the stage problem exists for initial state $x(0)$, then the controller is both stable and persistently feasible.

- setting $\mathcal{X}_f = \{0\}$ may drastically reduce the set states for which the stage optimization problem is feasible
- if a solution does exist it might be very costly
Terminal cost and Terminal Constraints

**Obvious:** If $\mathcal{X}_f = \{0\}$ and a solution to the stage problem exists for initial state $x(0)$, then the controller is both stable and persistently feasible.

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**Definition (Region of Attraction)**

Given a controller $u_k = f(x_k)$, its *region of attraction* $\mathcal{X}_0$ is

$$\mathcal{X}_0 = \{x_0 \in \mathcal{X} : \lim_{k \to \infty} x_k = 0, x_k \in \mathcal{X}, f(x_k) \in \mathcal{U}, \forall k\}$$

where $x_{j+1} = Ax_j + B f(x_j)$ for each $j$

**Goal:** Find a tradeoff between

- control performance (minimize cost)
- size of the region of attraction (can control larger set of states)
- computational complexity (grows with the prediction horizon)
Region of Attraction

Recall: Double Integrator Example

\[
A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad R = 10
\]

\[\mathcal{X} = \{ x \in \mathbb{R}^2 : |x_i| \leq 5, \; i = 1, 2 \}, \quad \mathcal{U} = \{ u \in \mathbb{R} : |u| \leq 1 \}\]
Region of Attraction

Recall: Double Integrator Example

\[
A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad R = 10
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6. Extensions and Generalizations
Some Set-Theoretic Notions

**Definition (Invariant Set)**

Given an autonomous system \( x_{k+1} = f(x_k) \), a set \( \mathcal{O} \subset \mathcal{X} \) is *(positively) invariant* if

\[
x_0 \in \mathcal{O} \quad \Rightarrow \quad x_k \in \mathcal{O} \quad \text{for all} \quad k \geq 0
\]
Some Set-Theoretic Notions

Definition (Invariant Set)

Given an autonomous system $x_{k+1} = f(x_k)$, a set $\mathcal{O} \subset \mathcal{X}$ is (positively) invariant if

$$x_0 \in \mathcal{O} \quad \Rightarrow \quad x_k \in \mathcal{O} \text{ for all } k \geq 0$$

Definition (Maximal Invariant Set)

The set $\mathcal{O}_\infty$ is the maximal invariant set if $0 \in \mathcal{O}_\infty$, $\mathcal{O}_\infty$ is invariant and $\mathcal{O}_\infty$ contains all invariant sets that contain the origin.
### Some Set-Theoretic Notions

#### Definition (Invariant Set)

Given an autonomous system \( x_{k+1} = f(x_k) \), a set \( O \subset \mathcal{X} \) is (positively) invariant if

\[
x_0 \in O \implies x_k \in O \text{ for all } k \geq 0
\]

#### Definition (Maximal Invariant Set)

The set \( O_\infty \) is the maximal invariant set if \( 0 \in O_\infty \), \( O_\infty \) is invariant and \( O_\infty \) contains all invariant sets that contain the origin.

#### Definition (Control Invariant Set)

A set \( C \subset \mathcal{X} \) is said to be control invariant if

\[
x_0 \in C \implies \exists (u_0, u_1, \ldots) \in \mathcal{U}_\infty \text{ s.t. } x_k \in C \text{ for all } k \geq 0
\]

where \( x_{j+1} = Ax_j + Bu_j \) for all \( j \)
Ensuring Feasibility

**Task:** We must make sure that, in each step, there exists a feasible control from the predicted state $x_N$.

- can achieve this if $\mathcal{X}_f = \mathcal{C}$ for some control invariant set $\mathcal{C}$
- idea: use the maximal invariant set of a particular controller
Ensuring Feasibility

**Task:** We must make sure that, in each step, there exists a feasible control from the predicted state $x_N$.

- can achieve this if $X_f = C$ for some control invariant set $C$
- idea: use the maximal invariant set of a particular controller

**Definition (Maximal LQR Invariant Set)**

The *maximal LQR invariant set* $O_{\infty}^{LQR}$ is the maximal invariant set of the system $x_{k+1} = (A - BK)x_k$ subject to the constraint $u_k = -Kx_k \in \mathcal{U}$, where $K$ is the LQR gain.

**Terminal Constraint:**

- if $X_f = O_{\infty}^{LQR}$, then we are guaranteed to be persistently feasible (optimization can always choose LQR control, feasible by definition)
- determining $O_{\infty}^{LQR}$ is computationally feasible in some cases (but problematic in high dimensions with complicated constraint sets)
Ensuring Stability

Basic Idea:
Proof is based on a Lyapunov argument
- standard method in nonlinear control
- based on showing that a Lyapunov function, an energy-like function of the state, is decreasing along closed-loop system trajectories
- details outside the scope of this class (see EE222, ME237 for details)
Ensuring Stability

**Basic Idea:**
Proof is based on a *Lyapunov* argument

- standard method in nonlinear control
- based on showing that a *Lyapunov function*, an energy-like function of the state, is decreasing along closed-loop system trajectories
- details outside the scope of this class (see EE222, ME237 for details)

**How to ensure this:**

- terminal weighting matrix $P$ commonly chosen as the matrix solving the *Discrete-Time Algebraic Riccati Equation*:

\[
P = A^T PA + Q - A^T PB (B^T PB + R)^{-1} B^T PA
\]

- $x_k^T P x_k$ is the infinite horizon cost of regulating the system state to zero starting from $x_k$, using the (unconstrained) LQR controller

**Note:** $x_N \in O_{\infty}^{LQR} \Rightarrow$ starting from $x_N$ LQR is feasible (hence optimal)
Main Theoretical Result

**Theorem (Stability and Persistent Feasibility for Basic MPC)**

*Suppose that*

1. \( Q = Q^T \succ 0, \ R = R^T \succ 0 \) and \( P \succ 0 \)
2. \( \mathcal{X}, \mathcal{X}_f \) and \( \mathcal{U} \) are closed sets containing the origin in their interior
3. \( \mathcal{X}_f \) is control invariant, \( \mathcal{X}_f \subseteq \mathcal{X} \)
4. \( \min_{u \in \mathcal{U}, x^+ \in \mathcal{X}_f} -x^T P x + x^T Q x + u^T R u + (x^+)^T P (x^+) \leq 0 \)
   for all \( x \in \mathcal{X}_f \), where \( x^+ = Ax + Bu \)

*Then*

1. the state of the closed-loop system converges to the origin, i.e. \( \lim_{k \to \infty} x(k) = 0 \)
2. the origin of the closed-loop system is asymptotically stable with domain of attraction \( \mathcal{X}_0 \)
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**Modified Double Integrator Example**

\[
A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad R = 10
\]

\[X = \{ x \in \mathbb{R}^2 : |x_i| \leq 5, \ i = 1, 2 \}\]

\[U = \{ u \in \mathbb{R} : |u| \leq 1 \}\]

\[X_f = O_{\infty}^{LQR}\]

\[P \text{ solves DARE}\]
Example: Constrained Double Integrator

Simulation Results

Observations

- controller with $N = 3$ is infeasible at $t = 0$
- same performance for controllers with $N = 5$ and $N = 10$
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Extensions and Generalizations

The basic ideas of MPC can be extended to

- MPC for reference tracking
- Robust MPC (robustness w.r.t disturbances / model errors)
- Output-feedback MPC (state only partially observed)
- Fast MPC for applications with fast dynamics
  - fast solvers for online MPC
  - explicit MPC via multiparametric programming
- MPC for nonlinear systems
- ...
**Example: Something of Everything**

*Interpolated Tube-Based Robust Output Feedback MPC for Tracking*

- output-feedback
- robust to bounded additive disturbances
- set-point tracking
- interpolation between different terminal constraint sets
  - allows to improve feasibility while maintaining high performance
Explicit Model Predictive Control

Solving the Quadratic Program via Multiparametric Programming

- the stage quadratic program can be solved explicitly
- solution is a piece-wise affine control law
- region of attraction partitioned into polyhedra ⇒ Lookup-Table
Model Predictive Control is

- a very active research area
- becoming a mature technology
- applicable to a wide range of control problems
- the only control methodology taking hard state and input constraints into account explicitly

Model Predictive Control is not

- “plug and play” – it requires specialized knowledge
- inherently “better” than other control techniques
- widely accepted in industry (yet)
Want to Learn More?

**Recommended Classes**
- EE 127: Optimization Models in Engineering (El Ghaoui)
- ME 190M: Introduction to Model Predictive Control (Borrelli)

**Advanced Classes**
- EE 221A: Linear System Theory
- EE 227B: Convex Optimization (El Ghaoui)
- ME 290J: Model Predictive Control for Linear and Hybrid Systems (Borrelli)
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